MATH5010 Linear Analysis (2022-23): Homework 6. Deadline: 13 Nov 2022

Important Notice:

♣ The answer paper must be submitted before the deadline.
♠ The answer paper MUST BE sent to the CU Blackboard.

1. Let X and Y be the normed spaces over $\mathbb{R}$. For each element $(x, y) \in X \times Y$, define the norm by $q(x, y) := \max(\|x\|, \|y\|)$. (Recall: $(x, y) + (x', y') := (x + x', y + y')$ and $\alpha(x, y) := (\alpha x, \alpha y)$ for $(x, y), (x', y') \in X \times Y$ and $\alpha \in \mathbb{R}$.) Let $\pi : X \times Y \to X$ be a linear map defined by $\pi(x, y) := x$ for $(x, y) \in X \times Y$.
   Show that the map π is bounded and find the norm $\|\pi\|$.

2. Let $\ell_2 := \{x : \{1, 2...\} \to \mathbb{R} : \sum |x(n)|^2 < \infty\}$ and put $\|x\|_2 := \sqrt{\sum |x(n)|^2}$. Let $X := \{x \in \ell_2 : \sum_{n=1}^{\infty} |nx(n)|^2 < \infty\}$. Define a linear operator $T : X \to \ell_2$ by

$$Tx(n) := nx(n) \quad \text{for } x \in X \text{ and } n = 1, 2, ...$$

   (i) Is T a bounded operator? (Explain !)

   (ii) Show that the inverse $T^{-1} : \ell_2 \to X$ is bounded and find $\|T^{-1}\|$.

$* * *$ **End** $* * *$